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On Curvature Expansion of Higher Spin Gauge Theory

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Abstract

We examine the curvature expansion of a the field equations of a four–dimensional higher spin gauge theory extension of anti-de Sitter gravity. The theory contains massless particles of spin $0, 2, 4, ...$ that arise in the symmetric product of two spin 0 singletons. We cast the curvature expansion into manifestly covariant form and elucidate the structure of the equations and observe a significant simplification.

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1 Introduction

It is reasonable to assume that the interactions of quantum gravity simplifies in the limit of high energies, such that only a limited spectrum survives whose interactions are governed by a symmetry group, analogously to how supergravity/supersymmetry emerges in the low energy limit. Clearly, we do not expect this theory to be another supergravity theory, but instead it seems much more suggestive to consider some massless higher spin extension of supergravity. These are gauge theories based on infinite dimensional algebras which are essentially given by the enveloping algebras of an underlying anti-de Sitter superalgebra. Thus the resulting higher spin gauge theory is an extension of the corresponding gauged supergravity theory, capable of interpolating between supergravity at low energies and its higher spin extension at high energies.

The higher spin gauge theories in four dimensions have been primarily developed by Vasiliev [1]. For reviews, see [2, 3]. In this report we shall describe some of the basic properties of the higher spin theories in the context of an ordinary nonsupersymmetric algebra [4]. We also examine the curvature expansion of the field equations and write these on a manifestly covariant form (i.e. without reference to the special anti-de Sitter solution). In particular, we point at a cancellation of certain structures at higher orders (beginning at the second order) that leads to a significant simplification of the higher spin equations. The covariant form of the action for the physical gauge fields, but with the lower spin sector set equal to zero, was given up to cubics in curvatures already in [5], while the covariant field equations given here include all physical fields as well as auxiliary gauge fields and are valid to arbitrary order in curvatures.

The paper is organized as follows. In Section 2 we review the formulation of anti-de Sitter gravity in the constraint formalism, the bosonic higher spin algebra and its unitary representation on a spin zero singleton, and give the corresponding basic field content of the higher spin gauge theory. With these preliminaries, we then discuss Vasiliev's procedure for constructing interactions in Section 3. In Section 4 we examine the resulting covariant curvature expansion of the higher spin equations. In Section 5 we conclude and describe briefly work in progress based on the results on Section 4.

2 Preliminaries

Our starting point is the $D = 4$ Einstein's equation with a negative cosmological constant:

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad , \qquad \Lambda < 0 \tag{2.1}$$

Viewed as a curvature constraint it leaves ten components of the Riemann tensor $R_{\mu\nu,\lambda\rho}$ unconstrained. These form an irreducible tensor called the Weyl curvature tensor. In van der Waerden notation [1] this tensor is $C_{\alpha\beta\gamma\delta}$. We can write (2.1) in an equivalent first order form as follows:

[1] An $SO(3,1)$ vector $V_a = (\sigma_a)^{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}}$ where α and its hermitian conjugate $\dot{\alpha}$ are two component indices raised and lowered by $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ using NE-SW see–saw rules for both dotted and undotted indices.

$$\mathcal{R}_{\mu\nu,ab} = e_\mu{}^c e_\nu{}^d (\sigma_{cd})^{\alpha\beta}(\sigma_{ab})^{\gamma\delta} C_{\alpha\beta\gamma\delta} + \text{h.c.} \ , \tag{2.2}$$

$$\mathcal{R}_{\mu\nu,a} = 0 \ , \tag{2.3}$$

where $\mathcal{R} = d\omega + g\omega\wedge\omega = \frac{1}{2i}(\mathcal{R}^{ab}M_{ab} + 2\mathcal{R}^a P_a)$ is $SO(3,2)$ valued and g is gauge coupling. The $SO(3,2)$ gauge field has components $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a = \sqrt{2}\kappa^{-1}e_\mu{}^a$, that we identify with the Lorentz connection and the vierbein, respectively. Here κ^2 is the 4D Newton's constant. The Lorentz valued curvature is related to the ordinary Riemann tensor through

$$\mathcal{R}_{\mu\nu,ab} = R_{\mu\nu,ab} + 4g\kappa^{-2}e_\mu{}^{[a}e_\nu{}^{b]} \ , \tag{2.4}$$

while the $\mathcal{R}_{\mu\nu,a}$ is the usual torsion. Tracing (2.2) with $e^{\nu,b}$ gives $\mathcal{R}_{\mu a} = 0$, which together with (2.3) yields (2.1) with the identification

$$\Lambda = -\frac{6g^2}{\kappa^2} \ . \tag{2.5}$$

In order to extend $SO(3,2)$ to a higher spin algebra we start from the following realization of $SO(3,2)$ in terms of Grassmann even $SO(3,2)$ spinors:

$$M_{ab} = \tfrac{1}{4}(\sigma_{ab})^{\alpha\beta} y_\alpha y_\beta + \tfrac{1}{4}(\bar\sigma_{ab})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta} \ , \tag{2.6}$$

$$P_a = \tfrac{1}{2}(\sigma_a)^{\alpha\dot\alpha} y_\alpha \bar y_{\dot\alpha} \ , \tag{2.7}$$

where the spinors satisfy the following oscillator algebra [2]

$$y_\alpha \star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta} \ , \quad \bar y_{\dot\alpha}\star\bar y_{\dot\beta} = \bar y_{\dot\alpha}\bar y_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta} \ , \tag{2.8}$$

$$y_\alpha \star \bar y_{\dot\beta} = y_\alpha \bar y_{\dot\beta} \ , \qquad \bar y_{\dot\alpha}\star y_\beta = \bar y_{\dot\alpha} y_\beta \ . \tag{2.9}$$

Here $\star$ denotes the operator product and the products without $\star$ are Weyl ordered (i.e. totally symmetric) products. The $\star$ product of two arbitrary functions of y and $\bar y$ is given by

$$F\star G = \int d^4u\ d^4v\ F(y+u,\bar y+\bar u)\ G(y+v,\bar y+\bar v)\ e^{i(u_\alpha v^\alpha + \bar u_{\dot\alpha}\bar v^{\dot\alpha})} \ . \tag{2.10}$$

An irreducible highest weight representation $D(E,s)$ of $SO(3,2)$ is labeled by the energy E and spin s of its ground state, where the energy and spin operators are M_{05} and M_{12}, respectively.

[2] The spinor $Y_{\underline\alpha} \equiv (y_\alpha, \bar y_{\dot\alpha})$ is a Majorana spinor of $S0(3,2)\simeq Sp(4)$. The algebras (2.8-2.9) and (3.1-3.4) can be written in a manifestly $Sp(4)$ invariant form; e.g. $Y_{\underline\alpha}\star Y_{\underline\beta} = Y_{\underline\alpha}Y_{\underline\beta} + C_{\underline{\alpha\beta}}$.

Unitarity requires $E \geq s+1$, $(E,s) = (\frac{1}{2},0)$ or $(E,s) = (1,\frac{1}{2})$. In the case that $s \geq 1$, there are extra null-states for $E = s+1$, and $D(s+1,s)$, $s \geq 1$, are thus referred to as massless representations. The scalar $D(1,0)$, the pseudo–scalar $D(2,0)$ and the spin–$\frac{1}{2}$ representation $D(\frac{3}{2},\frac{1}{2})$ are also referred to as massless representations. The two special cases $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$ have only a finite number of states of any given spin and cannot propagate in four dimensions. These are the Rac and Di singletons, whose dynamics is restricted to the conformal boundary of AdS. The states in the Fock space of the oscillators (2.8-2.9) with even and odd occupation number generate the weight space of $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$, respectively.

In this report we shall consider a four–dimensional bosonic higher spin extension of $SO(3,2)$ obtained from the algebra of polynomials $P(y,\bar{y})$ in y_α and $\bar{y}_{\dot{\alpha}}$ modulo the following projection and reality conditions:

$$\tau(P(y,\bar{y})) \equiv P(iy,i\bar{y}) = -P \quad , \qquad P^\dagger = -P \ . \tag{2.11}$$

These conditions define the Lie algebra $hs_2(1)$ [4] with respect to the bracket $[P,Q] = P \star Q - Q \star P$, since τ and the $\dagger$ have the properties:

$$\tau(P \star Q) = \tau(Q) \star \tau(P) \ , \quad (P \star Q)^\dagger = Q^\dagger \star P^\dagger \ . \tag{2.12}$$

The algebra $hs_2(1)$ is a direct sum of spaces of monomials in y and $\bar{y}$ of degree $2,6,10,\ldots$. We use a notation such that if P is an analytical function of y and $\bar{y}$ then

$$P_{\alpha(k)\dot{\alpha}(l)} = \frac{1}{k!l!}\partial_{\alpha_1}\cdots\partial_{\alpha_k}\bar{\partial}_{\dot{\alpha}_1}\cdots\bar{\partial}_{\dot{\alpha}_l}P|_{Y=0} \ . \tag{2.13}$$

The space of bilinears of $hs_2(1)$ is isomorphic to $SO(3,2)$, which is the maximal finite subalgebra of $hs_2(1)$.

From the above considerations it follows that $hs_2(1)$ can be represented unitarily on $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$. This immediately yields a three-dimensional realization of $hs_2(1)$ as a current algebra constructed from the singleton free field theory. A four–dimensional field theory realization of $hs_2(1)$ must be based on a UIR of $hs_2(1)$ that decompose into $SO(3,2)$ UIR's with $E \geq s+1$. Such a UIR is given by the symmetric tensor product

$$\Big(D(\tfrac{1}{2},0) \otimes D(\tfrac{1}{2},0)\Big)_S = D(1,0) \oplus D(3,2) \oplus D(5,4) \cdots \ , \tag{2.14}$$

which corresponds to a scalar, a graviton and a tower of massless higher spin fields with spins $4,6,\ldots$. Note that the spin $s \leq 2$ sector of this spectrum contains a single real scalar and as such it can not correspond to a bosonic subsector of a (matter coupled) higher spin supergravity theory. Nonetheless the scalar field is needed for unitary realization of the higher spin $hs_2(1)$ symmetry and thus, even in the bosonic higher spin theory, the field content is not arbitrary but rather is restricted in an interesting way.

In fact, a spectrum of states with the same spin content can also be obtained from the antisymmetric product of the fermionic singletons $D(1,\frac{1}{2})$, the difference being that the scalar field has lowest energy $E_0 = 2$ rather than $E_0 = 1$, which is the case in (2.14).

It is worthwhile to note that the oscillator algebra has four (linear) anti-involutions [4]: τ, $\tau\pi$, $\tau\bar{\pi}$ and $\tau\pi\bar{\pi}$, where π and $\bar{\pi}$ are defined below in (2.20). Projecting by τ and $\tau\pi\bar{\pi}$ leads to $hs_2(1)$, while $\tau\pi$ and $\tau\bar{\pi}$ leads to higher spin algebras that do not contain the translations. The oscillator algebra also has an involution $\rho \equiv \tau^2 = \pi\bar{\pi}$. Projecting by imposing $\rho(P) = P$, gives rise to a reducible higher spin algebra $hs(1)$ [4] with spins $1, 2, 3, \ldots$. The gauging of this algebra gives rise to a spectrum given by $D(\frac{1}{2}, 0) \otimes D(\frac{1}{2}, 0)$.

To construct a four–dimensional field theory with symmetry algebra $hs_2(1)$ and spectrum (2.14) one needs to introduce an $hs_2(1)$ valued gauge field $\mathcal{A}_\mu$ *and* a scalar master field Φ in a representation of $hs_2(1)$ containing the physical scalar ϕ, the spin 2 Weyl tensor $C_{\alpha\beta\gamma\delta}$, its higher spin generalizations $C_{\alpha(4n)}$ $n = 2, \ldots$ [3], and all the higher derivatives of these fields [1]. Let us first give an intuitive explanation of this, before we give the formal construction of the theory. The dynamics for the gauge fields follows from a curvature constraint of the form

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} \ , \qquad \mathcal{F} \equiv d\mathcal{A} + g\mathcal{A} \star \mathcal{A} \tag{2.15}$$

where two–form B is a function of $\mathcal{A}_\mu$ and Φ. It is assumed that the structure of (2.15) is analogous to (2.2), such that the Weyl tensors are given in terms of curvatures [5]. In fact, it has been shown by Vasiliev (see [2] for a review) that the spectrum (2.14) requires that the linearized expression for B (in a Φ expansion) must obey

$$B_{\alpha(m)\dot{\alpha}(n)} = \delta_{n0} e^a \wedge e^b (\sigma_{ab})^{\beta(2)} C_{\alpha(m-2)\beta(2)} - h.c. \ , \quad m + n = 2, 6, 10, \ldots \tag{2.16}$$

where we have expanded B, using the notation (2.13). From (2.15) it follows that $dB + g\mathcal{A} \star B - gB \star \mathcal{A} = 0$, and there are no further constraints. Thus, the master field Φ must contain not only the scalar field and the Weyl tensors, but also all their higher derivatives. The linearized scalar field equation can also be written in first order form as follows:

$$\partial_{\alpha\dot{\alpha}}\phi = ig\kappa^{-1}\Phi_{\alpha\dot{\alpha}} \ , \quad \nabla_{\alpha\dot{\alpha}}\Phi_{\beta\dot{\beta}} = ig\kappa^{-1}(\Phi_{\alpha\dot{\alpha}\beta\dot{\beta}} - \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\phi) \ . \tag{2.17}$$

Thus we find that the scalar field ϕ, the generalized Weyl tensors $C_{\alpha(4n)}$ and all their covariant derivatives fit into Φ as follows:

$$\begin{aligned} \Phi|_{y=0} &= \phi \ , \\ \Phi_{\alpha\dot{\alpha}} &= -ig^{-1}\kappa\partial_{\alpha\dot{\alpha}}\phi + \cdots \ , \\ \Phi_{\alpha(2)\dot{\alpha}(2)} &= i^2 g^{-2}\kappa^2 \partial_{\alpha_1\dot{\alpha}_1}\partial_{\alpha_2\dot{\alpha}_2}\phi + \cdots \ , \end{aligned}$$

[3] We use the notation $\alpha(n) \equiv (\alpha_1 \cdots \alpha_n)$. All spinor indices denoted by the same letter (with different subscripts) are assumed to be symmetrized with unit strength.

$$
\begin{aligned}
&\vdots \\
\Phi_{\alpha(4n)} &= g^{-2}\kappa^2 C_{\alpha(4n)} \ , \\
\Phi_{\alpha(4n)\beta\dot\beta} &= -ig^{-1}\kappa\partial_{\beta\dot\beta}C_{\alpha(4n)} + \cdots \ , \\
\Phi_{\alpha(4n)\beta(2)\dot\beta(2)} &= i^2 g^{-2}\kappa^2\partial_{\beta_1\dot\beta_1}\partial_{\beta_2\dot\beta_2}C_{\alpha(4n)} + \cdots \ , \\
&\vdots
\end{aligned}
\tag{2.18}
$$

and the hermitian conjugates, where $n = 1, 2, \ldots$, we have used the notation in (2.13) and the dots denote the covariantizations. Thus, the nonzero components of Φ are $\Phi_{\alpha(m)\dot\alpha(n)}$, $|m-n| = 0$ mod 4. This is equivalent to imposing the conditions:

$$
\tau(\Phi) = \bar\pi(\Phi) \ , \qquad \Phi^\dagger = \pi(\Phi) \ , \tag{2.19}
$$

where

$$
\pi(\Phi(y,\bar y)) = \Phi(-y,\bar y) \ , \qquad \bar\pi(\Phi(y,\bar y)) = \Phi(y,-\bar y) \ . \tag{2.20}
$$

This defines a 'quasi-adjoint' representation of $hs_2(1)$ with covariant derivative

$$
\mathcal{D}\Phi = d\Phi + g\mathcal{A}\star\Phi - g\Phi\star\bar\pi(\mathcal{A}) \ . \tag{2.21}
$$

Thus, the integrability condition on B_2 and the scalar field equation (2.17) must combine into a single constraint of the form $\mathcal{D}_\mu\Phi = B_\mu$, where the one-form B is a function of $\mathcal{A}_\mu$ and Φ. In summary, the higher spin field equations are given by the constraints

$$
\mathcal{F}_{\mu\nu} = B_{\mu\nu}(\mathcal{A},\Phi) \ , \qquad \mathcal{D}_\mu\Phi = B_\mu(\mathcal{A},\Phi) \ . \tag{2.22}
$$

3 Construction of the Constraints

In order to construct the interactions in $B_{\mu\nu}$ and B_μ one may employ a Noether procedure in which $d^2 = 0$ is satisfied order by order in an expansion in Φ (counted by powers of g). This can be facilitated by a geometrical construction based on extending the ordinary four–dimensional spacetime by an internal four–dimensional noncommutative space with spinorial coordinates z_α and $\bar z_{\dot\alpha}$ obeying the basic 'contraction rules'[1, 2]

$$
z_\alpha\star z_\beta = z_\alpha z_\beta - i\epsilon_{\alpha\beta} \ , \quad z_\alpha\star y_\beta = z_\alpha y_\beta + i\epsilon_{\alpha\beta} \ , \tag{3.1}
$$

$$
y_\alpha\star z_\beta = y_\alpha z_\beta - i\epsilon_{\alpha\beta} \ , \quad y_\alpha\star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta} \ , \tag{3.2}
$$

$$\bar{z}_{\dot{\alpha}} \star \bar{z}_{\dot{\beta}} = \bar{z}_{\dot{\alpha}}\bar{z}_{\dot{\beta}} - i\epsilon_{\dot{\alpha}\dot{\beta}} , \quad \bar{z}_{\dot{\alpha}} \star \bar{y}_{\dot{\beta}} = \bar{z}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} - i\epsilon_{\dot{\alpha}\dot{\beta}} , \tag{3.3}$$

$$\bar{y}_{\dot{\alpha}} \star \bar{z}_{\dot{\beta}} = \bar{y}_{\dot{\alpha}}\bar{z}_{\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}} , \quad \bar{y}_{\dot{\alpha}} \star \bar{y}_{\dot{\beta}} = \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}} , \tag{3.4}$$

together with $z \star \bar{z} = z\bar{z}$, $\bar{z} \star z = \bar{z}z$, $z \star \bar{y} = z\bar{y}$ and $\bar{z} \star y = \bar{z}y$, with the following generalization to arbitrary functions of $(y, \bar{y})$ and $(z, \bar{z})$:

$$F \star G = \int d^4u \ d^4v \ F(y+u, \bar{y}+\bar{u}; z+u, z-\bar{u}) \ G(y+v, \bar{y}+\bar{v}; z-v, \bar{z}+\bar{v}) \ e^{i(u_\alpha v^\alpha + \bar{u}_{\dot{\alpha}}\bar{v}^{\dot{\alpha}})} \ . \tag{3.5}$$

The above (associative) algebra is equivalent to the normal ordered product of a pair of symplectic oscillators. In the extended spacetime, one considers an integrable system consisting of a one–form $\hat{\mathcal{A}} = dx^\mu \hat{\mathcal{A}}_\mu + dz^\alpha \hat{\mathcal{A}}_\alpha + d\bar{z}^{\dot{\alpha}}\hat{\mathcal{A}}_{\dot{\alpha}}$ and scalar $\hat{\Phi}$ defined by [2]

$$\tau(\hat{\mathcal{A}}) = -\hat{\mathcal{A}} , \qquad \hat{\mathcal{A}}^\dagger = \hat{\mathcal{A}} , \tag{3.6}$$

$$\tau(\hat{\Phi}) = \bar{\pi}(\hat{\Phi}) , \qquad \hat{\Phi}^\dagger = \pi(\hat{\Phi}) , \tag{3.7}$$

where the anti-involution τ and the involutions π and $\bar{\pi}$ have been extended as follows:

$$\tau(f(y,\bar{y};z,\bar{z})) = f(iy, i\bar{y}; -iz, -i\bar{z}) , \tag{3.8}$$

$$\pi(f(y,\bar{y};z,\bar{z})) = f(-y, \bar{y}; -z, \bar{z}) , \tag{3.9}$$

$$\bar{\pi}(f(y,\bar{y};z,\bar{z})) = f(y, -\bar{y}; z, -\bar{z}) . \tag{3.10}$$

By definition the exterior derivative $\hat{d}$ commutes with the maps in (3.8-3.10), such that $\tau(dz^\alpha) = -idz^\alpha$, $\pi(dz^\alpha) = -dz^\alpha$ and $\bar{\pi}(dz^\alpha) = dz^\alpha$.

The concise form of the full higher spin field equations was first given in [1]. As emphasized in [6] these equations are equivalent to the following curvature constraint:

$$\hat{\mathcal{F}} \equiv \hat{d}\hat{\mathcal{A}} + g\hat{\mathcal{A}} \star \hat{\mathcal{A}} = \tfrac{i}{4} dz^\alpha \wedge dz_\alpha \hat{\Phi} \star \kappa + \tfrac{i}{4} d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}} \hat{\Phi} \star \bar{\kappa} , \tag{3.11}$$

where the element κ is defined by

$$\kappa = \exp iz_\alpha y^\alpha , \quad \bar{\kappa} \equiv \kappa^\dagger = \exp -i\bar{z}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}} . \tag{3.12}$$

Multiplication by κ connects the two representations given in (3.6-3.7), which follows from using (2.12) and the following basic lemmas:

$$\kappa \star f(y,\bar{y};z,\bar{z}) = \kappa f(z,\bar{y};y,\bar{z}) , \qquad f(y,\bar{y};z,\bar{z}) \star \kappa = \kappa f(-z,\bar{y};-y,\bar{z}) . \tag{3.13}$$

This property is crucial for the whole construction, as we shall see below. The Bianchi identity implies that Φ must satisfy

$$\hat{\mathcal{D}}\hat{\Phi} \equiv \hat{d}\hat{\Phi} + g\hat{\mathcal{A}} \star \hat{\Phi} - g\hat{\Phi} \star \bar{\pi}(\hat{\mathcal{A}}) = 0 \ . \tag{3.14}$$

The rationale behind (3.11) is the following:

1) From the $\alpha\beta$, $\alpha\dot{\beta}$, $\alpha\mu$ and $\dot{\alpha}\mu$ components of (3.11) and the α and $\dot{\alpha}$ components of (3.14), we can solve for the z and $\bar{z}$ dependence of all the fields in terms of the 'initial' conditions:

$$\hat{\mathcal{A}}_\mu|_{Z=0} = \mathcal{A}_\mu \ , \qquad \hat{\Phi}|_{Z=0} = \Phi \ . \tag{3.15}$$

The integrability of (3.11) and (3.14) then implies that the remaining $\mu\nu$ and μ components of (3.11) and (3.14) are satisfied for all z_α and $\bar{z}_{\dot{\alpha}}$ provided that they are satisfied at $z_\alpha = \bar{z}_{\dot{\alpha}} = 0$, that is

$$\hat{\mathcal{F}}_{\mu\nu}|_{Z=0} = 0 \ , \qquad \hat{\mathcal{D}}_\mu\hat{\Phi}|_{Z=0} = 0 \ . \tag{3.16}$$

These are equations of the form (2.22), which by construction define an integrable set of constraints in ordinary spacetime.

2) Upon linearizing the constraints (3.11) and (3.14), it follows that $\hat{\Phi} = \Phi(y, \bar{y})$. From (3.13) it then follows that

$$\hat{\Phi} \star \kappa|_{Z=0} = \Phi(0, \bar{y}) \ , \qquad \hat{\Phi} \star \bar{\kappa}|_{Z=0} = \Phi(y, 0) \ . \tag{3.17}$$

Comparing with (2.18) one sees that the linearised two–form $B_{\mu\nu}$ in (2.22) depends on the the Weyl tensors $C_{\beta(2s)}$, $s = 2, 4, 6, ...$, but not on their derivatives, which is crucial in order for the linearized field equations to be of the right form (2.16).

3) Viewed as a Cartan integrable system, it is clear that Eqs. (3.11) and (3.14) are gauge invariant (and that spacetime diffeomorphisms are automatically incorporated into the gauge group). The gauge transformations leaving (3.11) and (3.14) invariant are

$$\delta\hat{\mathcal{A}} = d\hat{\epsilon} + g\hat{\mathcal{A}} \star \hat{\epsilon} - g\hat{\epsilon} \star \hat{\mathcal{A}} \ , \quad \delta\hat{\Phi} = g\hat{\epsilon} \star \hat{\Phi} - g\hat{\Phi} \star \pi(\hat{\mathcal{A}}) \ . \tag{3.18}$$

The Z–dependence in $\hat{\epsilon}$ can be used to impose the gauge condition

$$\hat{\mathcal{A}}_\alpha|_{Z=0} = 0 \ . \tag{3.19}$$

The gauge symmetries of the spacetime higher spin field equations (3.16) then become

$$\delta\mathcal{A}_\mu = \partial_\mu\epsilon + g(\hat{\mathcal{A}}_\mu \star \hat{\epsilon} - \hat{\epsilon} \star \hat{\mathcal{A}}_\mu)|_{Z=0} \ , \quad \delta\Phi = g(\hat{\epsilon} \star \hat{\Phi} - \hat{\Phi} \star \pi(\hat{\epsilon}))|_{Z=0} \ , \tag{3.20}$$

where the residual gauge transformations $\hat{\epsilon}$ are solved from

$$\partial_\alpha \hat{\epsilon} + g(\hat{\mathcal{A}}_\alpha \star \hat{\epsilon} - \hat{\epsilon} \star \hat{\mathcal{A}}_\alpha)|_{Z=0} = 0 \ , \quad \hat{\epsilon}|_{Z=0} = \epsilon \ . \tag{3.21}$$

and its hermitian conjugate, where the initial condition ϵ generates the original higher spin gauge algebra $hs_2(1)$.

4) Importantly, due to the mixing between Y and Z in (3.1-3.4), both (3.16) and (3.20) receive nontrivial corrections which 'deform' the spacetime curvatures such that (3.16) describe an interacting system.

4 The Covariant Curvature Expansion

Provided that $g << 1$, it makes sense to solve the spinorial components of the higher spin equations (3.11) and (3.14) subject to the initial condition (3.15) by expanding in Φ as follows:

$$\hat{\Phi} = \sum_{n=1}^{\infty} g^{n-1} \hat{\Phi}^{(n)} \ , \tag{4.1}$$

$$\hat{\mathcal{A}}_\alpha = \sum_{n=0}^{\infty} g^{n-1} \hat{\mathcal{A}}_\alpha^{(n)} \ , \tag{4.2}$$

$$\hat{\mathcal{A}}_\mu = \sum_{n=0}^{\infty} g^{n} \hat{\mathcal{A}}_\mu^{(n)} \ , \tag{4.3}$$

where the superscript n refers to terms that are n'th order in Φ. We begin by expanding the purely spinorial components of (3.11) as follows ($n \geq 0$):

$$\partial^\alpha \hat{\mathcal{A}}_\alpha^{(n)} = \tfrac{i}{2} \hat{\Phi}^{(n)} \star \kappa - \sum_{j=0}^{n} \hat{\mathcal{A}}^{(j)\alpha} \star \hat{\mathcal{A}}_\alpha^{(n-j)} \ , \tag{4.4}$$

$$\partial_\alpha \hat{\mathcal{A}}_{\dot{\alpha}}^{(n)} - \partial_{\dot{\alpha}} \hat{\mathcal{A}}_\alpha^{(n)} = \sum_{j=1}^{n-1} \left(\hat{\mathcal{A}}_{\dot{\alpha}}^{(j)} \star \hat{\mathcal{A}}_\alpha^{(n-j)} - \hat{\mathcal{A}}_\alpha^{(n-j)} \star \hat{\mathcal{A}}_{\dot{\alpha}}^{(j)} \right) \ , \tag{4.5}$$

and the spinorial components of (3.14) as follows ($n \geq 0$):

$$\partial_\alpha \hat{\Phi}^{(n)} = \sum_{j=1}^{n-1} \left(\hat{\Phi}^{(j)} \star \bar{\pi}(\hat{\mathcal{A}}_\alpha^{(n-j)}) - \hat{\mathcal{A}}_\alpha^{(n-j)} \star \hat{\Phi}^{(j)} \right) \ . \tag{4.6}$$

These equations form an integrable equation system for $\hat{\mathcal{A}}_\alpha^{(n)}$ $(n \geq 0)$ and $\hat{\Phi}^{(n)}$ $(n \geq 1)$ subject to the initial condition (3.15) that we solve by 'zig-zagging' back and forth between (4.4-4.5) and (4.6). For $n = 0$, eqs. (4.4-4.5) show that $\hat{\mathcal{A}}_\alpha^{(0)}$ is a gauge artifact. In the gauge (3.19), we can therefore set

$$\hat{\mathcal{A}}_\alpha^{(0)} = 0 \ . \tag{4.7}$$

We continue the zig-zagging by taking $n = 1$ in (4.6), whose right hand side also vanishes. The solution, satisfying the initial condition (3.15) is therefore

$$\hat{\Phi}^{(1)} = \Phi(y, \bar{y}) \ . \tag{4.8}$$

This result can then be used in (4.4-4.5) for $n = 1$ to solve for $\hat{\mathcal{A}}_\alpha^{(1)}$ as follows

$$\hat{\mathcal{A}}_\alpha^{(1)} = -\tfrac{i}{2} z_\alpha \int_0^1 tdt \ \hat{\Phi}(-tz, \bar{y}) \kappa(tz, y) \ . \tag{4.9}$$

The results (4.8) and (4.9) can then be used in (4.6) for $n = 2$ to solve for $\hat{\Phi}^{(2)}$, and so on. This generates the following series expansion $(n \geq 2)$:

$$\hat{\Phi}^{(n)} = z^\alpha \sum_{j=1}^{n-1} \int_0^1 dt \left(\hat{\Phi}^{(j)} \star \bar{\pi}(\hat{\mathcal{A}}_\alpha^{(n-j)}) - \hat{\mathcal{A}}_\alpha^{(n-j)} \star \hat{\Phi}^{(j)} \right)(tz, t\bar{z}) \quad + \ \text{h.c.} \ , \tag{4.10}$$

$$\begin{aligned} \hat{\mathcal{A}}_\alpha^{(n)} \quad &= \quad -z_\alpha \int_0^1 tdt \left(\tfrac{i}{2} \hat{\Phi}^{(n)} \star \kappa - \sum_{j=1}^{n-1} \hat{\mathcal{A}}^{(j)\beta} \star \hat{\mathcal{A}}_\beta^{(n-j)} \right)(tz, t\bar{z}) \\ &\quad + \bar{z}^{\dot{\beta}} \sum_{j=1}^{n-1} \int_0^1 tdt \left[\hat{\mathcal{A}}_{\dot{\beta}}^{(j)}, \hat{\mathcal{A}}_\alpha^{(n-j)} \right](tz, t\bar{z}) \ , \end{aligned} \tag{4.11}$$

where it is understood that the $\star$ products have to be evaluated before replacing $(z, \bar{z}) \to (tz, t\bar{z})$ and that the hermitian conjugate in (4.10) is in accordance with the reality condition $\hat{\Phi}^\dagger = \pi(\hat{\Phi})$.

Having analyzed the purely spinorial components of the constraint (3.11), we next analyze its $\alpha\mu$ components, which give the following equations for $\hat{\mathcal{A}}_\mu^{(n)}$ $(n \geq 0)$:

$$\partial_\alpha \hat{\mathcal{A}}_\mu^{(n)} = \partial_\mu \hat{\mathcal{A}}_\alpha^{(n)} + \sum_{j=1}^{n} \left[\hat{\mathcal{A}}_\mu^{(n-j)}, \hat{\mathcal{A}}_\alpha^{(j)} \right] \ . \tag{4.12}$$

These equations, together with the initial condition (3.15), can be integrated to yield

$$\hat{\mathcal{A}}^{(0)}_\mu = \mathcal{A}_\mu \ , \tag{4.13}$$

$$\hat{\mathcal{A}}^{(n)}_\mu = \int_0^1 dt \left[z^\alpha \left(\partial_\mu \hat{\mathcal{A}}^{(n)}_\alpha + \sum_{j=1}^{n} \left[\hat{\mathcal{A}}^{(n-j)}_\mu , \hat{\mathcal{A}}^{(j)}_\alpha \right] \right)(tz, t\bar{z}) \right.$$
$$\left. + \bar{z}^{\dot\alpha} \left(\partial_\mu \hat{\mathcal{A}}^{(n)}_{\dot\alpha} + \sum_{j=1}^{n} \left[\hat{\mathcal{A}}^{(n-j)}_\mu , \hat{\mathcal{A}}^{(j)}_{\dot\alpha} \right] \right)(tz, t\bar{z}) \right] \ . \tag{4.14}$$

Note that in (4.14) the second line is minus the hermitian conjugate of the first line. Substituting for $\hat{\mathcal{A}}^{(n)}_\alpha$ and $\hat{\mathcal{A}}^{(n)}_{\dot\alpha}$ by the expression given in (4.11), we observe that the first term in (4.11) drops out, using $z^\alpha z_\alpha = 0$. The second term, which is real, cancels against the same term coming from subtracting its hermitian conjugate, that is

$$z^\alpha \hat{\mathcal{A}}^{(n)}_\alpha + \bar{z}^{\dot\alpha} \hat{\mathcal{A}}^{(n)}_{\dot\alpha} = 0 \ . \tag{4.15}$$

This can be used to to simplify (4.14) further, with the result:

$$\hat{\mathcal{A}}^{(n)}_\mu = i \int_0^1 \frac{dt}{t} \left[\sum_{j=1}^{n} \left(\hat{\mathcal{A}}^{\alpha(j)} \star \partial^{(-)}_\alpha \hat{\mathcal{A}}^{(n-j)}_\mu + \partial^{(+)}_\alpha \hat{\mathcal{A}}^{(n-j)}_\mu \star \hat{\mathcal{A}}^{\alpha(j)} \right) \right.$$
$$\left. + \sum_{j=1}^{n} \left(\hat{\mathcal{A}}^{\dot\alpha(j)} \star \partial^{(+)}_\alpha \hat{\mathcal{A}}^{(n-j)}_\mu + \partial^{(-)}_{\dot\alpha} \hat{\mathcal{A}}^{(n-j)}_\mu \star \hat{\mathcal{A}}^{\dot\alpha(j)} \right) \right](tz, t\bar{z}) \ , \tag{4.16}$$

where $\partial^{(\pm)}_\alpha = \partial^{(z)}_\alpha \pm \partial^{(y)}_\alpha$. Finally, substituting (4.8), (4.10), (4.13) and (4.16) into the spacetime components of (3.11) and (3.14) one obtains an expansion of the constraints (2.22) as follows:

$$\mathcal{F}_{\mu\nu} = -\sum_{n=1}^{\infty} \sum_{j=0}^{n} g^{n+1} \left(\hat{\mathcal{A}}^{(j)}_{[\mu} \star \hat{\mathcal{A}}^{(n-j)}_{\nu]} \right) |_{Z=0} \ , \tag{4.17}$$

$$\mathcal{D}_\mu \Phi = \sum_{n=2}^{\infty} g^{n-1} \left(-\partial_\mu \hat\Phi^{(n)}|_{Z=0} + g \sum_{j=1}^{n} \left(\hat\Phi^{(j)} \star \bar\pi(\hat{\mathcal{A}}^{(n-j)}_\mu) - \hat{\mathcal{A}}^{(n-j)}_\mu \star \hat\Phi^{(j)} \right) |_{Z=0} \right) \ , \tag{4.18}$$

where the unhatted curvature $\mathcal{F}_{\mu\nu}$ and covariant derivative $\mathcal{D}_\mu \Phi$ are given in (2.15) and (2.21). The following comments are in order:

1) By construction the constraints (4.17-4.18) are integrable order by order in g, with $hs_2(1)$ gauge symmetry given by (3.20).

2) The curvatures $\mathcal{F}_{\mu\nu,\alpha(s)\dot{\alpha}(s-2)}$ contain the dynamical field equations for spin $s = 2, 4, 6, \ldots$. The remaining curvatures are generalized torsion equations, except for the pure curvatures $e_\mu{}^a e_\nu{}^b (\sigma_{ab})_{(\alpha_1\alpha_2}\mathcal{F}^{\mu\nu}{}_{,\alpha_3\ldots\alpha_{2s-2})}$ which are set equal to the generalized Weyl tensors $\Phi_{\alpha(2s)}$. The generalized torsion equations can be used to eliminate the auxiliary gauge fields $\mathcal{A}_{\mu,\alpha(m)\dot{\alpha}(n)}$, $|m-n| \geq 2$ in terms of the physical gauge fields $\mathcal{A}_{\mu,\alpha(s-1)\dot{\alpha}(s-1)}$.

3) The constraint (4.18) leads to the identifications in (2.18), and thus in particular contains the full version of the scalar field equation (2.17).

4) Setting $\mathcal{A}_\mu = \omega_\mu + W_\mu$, where ω_μ contain the $SO(3,2)$ gauge fields and W_μ the higher spin fields, the right hand side of (4.17) for $n = 1$, that is $-2g^2\mathcal{A}^{(0)}_{[\mu} \star \mathcal{A}^{(1)}_{\nu]}$, contains $\omega^2\Phi$ terms of the form (2.16). Thus, at the linear in Φ level, we can consistently set $W_\mu = \phi = 0$, which then yields the full, covariant Einstein equation (2.1) written in the form (2.2).

5 Comments

We have discussed how the constraint (3.11) in the extended spacetime gives rise to a manifestly covariant expansion of the higher spin field equations in terms of curvatures and a physical scalar field. In particular we have found a significant simplification in the interaction terms due to the identity (4.15). The resulting higher spin equations contain the full Einstein equation already at the leading order in the curvature expansion.

This raises the question of whether there exists any limit in which the higher spin equations reduce to ordinary anti-de Sitter gravity, possibly coupled to the scalar field ϕ. There are two natural parameters to scale in the problem: the gauge coupling g and a 'noncommutativity' parameter λ that can be introduced by taking $y \to \lambda y$ and $z \to \lambda z$. Scaling g is equivalent to scaling the derivatives ∂_μ, that is to taking a low energy, or weak curvature, limit, while scaling λ is equivalent to taking a Poisson limit of the higher spin algebra. In the latter limit the multicontractions are suppressed such that the commutator of a spin s and a spin s' generator closes on a spin $s + s' - 2$ generator.

One important issue to settle is whether the scalar ϕ disappears in this limit or if it stays in the spectrum, in which case the scalar potential should be computed from the expansion given above (in which case the term curvature expansion would of course strictly speaking be inappropriate). This issue is of extra interest in cases with extended supersymmetry since gauged supergravity requires scalars in particular cosets that determines the potential. In the nonsupersymmetric case a nontrivial potential could also be of interest, since it may cause the scalar to flow to other vacua than the anti-de Sitter vacuum at $\phi = 0$.

Other points of interest are, that in analogy with results for W-gravity in two dimensions [7], the Poisson limit could be considered as a mean for extracting a 'classical' higher spin theory whose quantization would then yield the full higher spin theory discussed here. There is also the issue of whether one can organize the expansion in curvatures such that it can be compared with the α' corrections coming from string theory, or M-theory. We will report elsewhere on some of the topics discussed above [8].

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